MIXED PLANE BOUNDARY VALUE PROBLEM OF THE THEORY OF ELASTICITY FOR A QUADRANT

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The problem can be reduced to an integral equation determining shear stresses at a clamped edge. The resulting solution makes it possible to supplement the results of investigation [1,2,3].

Let us study the stress problem in an elastic quadrant $x > 0$, $y > 0$ in the plane of variable $z = x + iy$ under the action of a concentrated force $Q + iP$, applied at the point $z_0 = x_0 + iy_0$ $(x_0 > 0,\ y_0 > 0)$. Let us assume that when $y = 0$ the displacements v, u are equal to zero, and when $x = 0$ the external loading are equal to zero (Fig. 1).

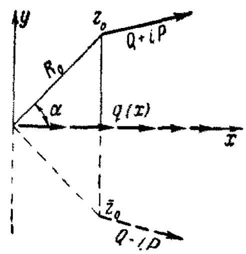

Fig. 1.

For the solution of the problem let us complete the quadrant to form a half-plane $x > 0$. Let us load symmetrically the new quadrant $x > 0$, $y < 0$ at the point $z_0 = x_0 - iy_0$ with a force $Q - iP$. Let us also introduce an additional, temporarily arbitrary loading $q(x)$ distributed along the x-axis. Evidently, under the action of symmetrical loadings $Q + iP$, $Q - iP$ and $q(x)$ on the half-plane $x > 0$ when $y = 0$, the displacement v is equal to zero. The loading $q(x)$ will be determined in such a way as to fulfil the condition $u = 0$ on the x-axis.

Let us study the state of stress of the given half-plane $x > 0$ with free edge $x = 0$ resulting from loadings $Q + iP$, $Q - iP$ and $q(x)$.

If for the stresses we make use of known representation,

$$X_x + Y_y = 2\,[\Phi(z) + \overline{\Phi(z)}]$$
$$Y_y - X_x + 2iX_y = 2\,[\bar{z}\Phi'(z) + \Psi(z)] \qquad (1)$$

then for a general case when the force $P + iQ$ is applied at the point $z_0 = x_0 + iy_0$ according to the formulas* of the paper [5] it is possible to obtain

$$\Phi_1(Q+iP,\ z,\ z_0) = -\frac{Q+iP}{2\pi(1+\varkappa)}\left(\frac{1}{z-z_0}+\frac{\varkappa}{z+\bar z_0}\right) - \frac{Q-iP}{2\pi(1+\varkappa)}\frac{z_0+\bar z_0}{(z+\bar z_0)^2} \tag{2}$$

$$\Psi_1(Q+iP,\ z,\ z_0) = \frac{Q-iP}{2\pi(1+\varkappa)}\left[\frac{\varkappa}{z-z_0}+\frac{1}{z+\bar z_0}+\frac{z_0+\bar z_0}{(z+\bar z_0)^2}\right] - \bar z_0\frac{d\Phi_1}{dz} \qquad \left(\varkappa = \frac{3-\nu}{1+\nu}\right)$$

In the case when loadings $Q + iP$, $Q - iP$ and $q(x)$ are acting on the half-plane, we will obtain

$$\Phi(z) = \Phi_1(Q+iP,\ z,\ z_0) + \Phi_1(Q-iP,\ z,\ \bar z_0) + \int_0^\infty \Phi_1[q(t),\ z,\ t]\,dt$$
$$\Psi(z) = \Psi_1(Q+iP,\ z,\ z_0) + \Psi_1(Q-iP,\ z,\ \bar z_0) + \int_0^\infty \Psi_1[q(t),\ z,\ t]\,dt \tag{3}$$

If $q(x)$ is determined from the condition $u = 0$ when $y = 0$, then the formulas (3) and (1) with $x > 0$, $y > 0$ will provide the solution of the problem for the stresses in an elastic quadrant with the assigned boundary conditions.

The condition $u = 0$ when $y = 0$, except for a rigid body displacement and taking into account that solution (3) satisfies the condition $v = 0$ when $y = 0$, is equivalent to the condition $u_x + iv_x = 0$. If representations (1) are made use of, the latter can be expressed as

$$\varkappa\Phi(x) - \overline{\Phi(x)} - x\overline{\Phi'(x)} - \overline{\Psi(x)} = 0 \tag{4}$$

Subjecting the functions $\Phi(z)$ and $\Psi(z)$ to be condition (4), we will obtain a singular integral equation for $q(x)$

$$2\varkappa\int_0^\infty \frac{q(t)}{t-x}\,dt - \int_0^\infty\left[\frac{1+\varkappa^2}{t+x}+\frac{4t(x-t)}{(t+x)^3}\right]q(t)\,dt = (Q+iP)F(x,\ z_0) + (Q-iP)F(x,\ \bar z_0) \tag{5}$$

where

$$F(x,\ z_0) = \frac{\varkappa}{x-z_0} + \frac{z_0-\bar z_0}{(x-z_0)^2} + \frac{\varkappa}{x-\bar z_0} + \frac{1}{x+z_0} + \frac{\varkappa(z_0+\bar z_0)}{(x+z_0)^2} - \frac{2(z_0-x)(z_0+\bar z_0)}{(x+z_0)^3} + \frac{\varkappa^2}{x+\bar z_0} - \frac{2\varkappa\bar z_0}{(x+\bar z_0)^2}$$

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* In deducing expression (2), an error was corrected in one of the formulas of paper [5].

Let us normalize equation (5), assuming

$$\frac{1}{\pi i}\int_0^\infty \frac{q(t)}{t-x}\,dt = \frac{r(x)}{\sqrt{x}} \tag{6}$$

With consideration of integrability of function $q(x)$, we have the transformation [4]

$$q(x) = \frac{1}{\pi i\sqrt{x}}\int_0^\infty \frac{r(t)}{t-x}\,dt \tag{7}$$

Introducing into equation (5) expressions (6) and (7) and changing the order of integration while taking into account that

$$\int_0^\infty \frac{dt}{\sqrt{t}\,(t+x)(t_1-t)} = \frac{\pi}{\sqrt{x}\,(x+t_1)}$$

$$\int_0^\infty \frac{t(x-t)\,dt}{\sqrt{t}\,(t_1-t)(t+x)^3} = -\frac{\pi\sqrt{x}}{4x(x+t_1)^3}\,(x^2-6xt_1+t_1^2)$$

we obtain the equation for the function $r(x)$

$$r(x) + \frac{\varkappa}{2\pi}\int_0^\infty \frac{r(t)}{x+t}\,dt + \frac{4}{\varkappa\pi}\int_0^\infty \frac{tx}{(t+x)^3}\,r(t)\,dt =$$
$$= \frac{\sqrt{x}}{2\varkappa\pi i}\left[(Q+iP)\,F(x,\,z_0) + (Q-iP)\,F(x,\,\bar{z}_0)\right] \tag{8}$$

Assuming that $t = e^\tau$, $x = e^\xi$, $r(x) = \psi(\xi)$, we can express equation (8) in the form

$$\psi(\xi) + \frac{\varkappa}{2\pi}\int_{-\infty}^\infty \frac{\psi(\tau)\,d\tau}{1+e^{\xi-\tau}} + \frac{4}{\varkappa\pi}\int_{-\infty}^\infty \frac{e^{\xi-\tau}}{(1+e^{\xi-\tau})^3}\,\psi(\tau)\,d\tau =$$
$$= \frac{\sqrt{e^\xi}}{2\varkappa\pi i}\left[(Q+iP)\,F(e^\xi,\,z_0) + (Q-iP)\,F(e^\xi,\,\bar{z}_0)\right] \tag{9}$$

Applying to both sides of the equation (9) the Laplace transform and using the notation

$$R(p) = \int_{-\infty}^\infty \psi(\xi)\,e^{-p\xi}\,d\xi$$

we obtain

$$R(p)\left[1 + \frac{\varkappa}{2\pi}\int_{-\infty}^\infty \frac{e^{-p\theta}\,d\theta}{1+e^\theta} + \frac{4}{\varkappa\pi}\int_{-\infty}^\infty \frac{e^{(1-p)\,\theta}\,d\theta}{(1+e^\theta)^3}\right] =$$
$$= \frac{1}{2\varkappa\pi i}\int_{-\infty}^\infty \left[(Q+iP)\,F(e^\xi,\,z_0) + (Q-iP)\,F(e^\xi,\,\bar{z}_0)\right] e^{(1/2-p)\,\xi}\,d\xi \tag{10}$$

For the integrals of the left and right sides of the equation we have

$$\int_{-\infty}^{\infty} \frac{e^{-p\theta}}{1+e^{\theta}} d\theta = -\frac{\pi}{\sin \pi p}, \qquad \int_{-\infty}^{\infty} \frac{e^{(1-p)\theta} d\theta}{(1+e^{\theta})^3} = \frac{p(p+1)}{2} \frac{\pi}{\sin \pi p}$$

$$\int_{-\infty}^{\infty} \frac{e^{(1/2-p)\xi}}{e^{\xi}-z_0} d\xi = \frac{\pi i e^{i\pi p}}{\cos \pi p} z_0^{-p-1/2}, \quad \int_{-\infty}^{\infty} \frac{e^{(1/2-p)\xi}}{(e^{\xi}-z_0)^2} d\xi = -\frac{\pi i e^{i\pi p}}{2\cos \pi p}(2p+1) z_0^{-p-3/2}$$

$$\int_{-\infty}^{\infty} \frac{e^{(1/2-p)\xi}}{(e^{\xi}-z_0)^3} d\xi = \frac{\pi i e^{i\pi p}}{8\cos \pi p}(2p+1)(2p+3) z_0^{-p-5/2}$$

Here $-1/2 < \mathrm{Re}\, p < 0$. We can now write equation (10) as

$$R(p)\left[1 - \frac{\varkappa}{2\sin \pi p} + \frac{2}{\varkappa}\frac{p(p+1)}{\sin \pi p}\right] =$$
$$= \frac{e^{\pi i p}}{2\varkappa \cos \pi p}\{(Q+iP)F_1(p, z_0) + (Q-iP)F_1(p, \bar{z}_0)\} \tag{11}$$

Here

$$F_1(p, z_0) = \varkappa z_0^{-p-1/2} - (z_0 - \bar{z}_0)\left(p+\frac{1}{2}\right) z_0^{-p-3/2} + \varkappa \bar{z}_0^{-p-\frac{1}{2}} + (-z_0)^{-p-1/2} -$$
$$-\varkappa(z_0+\bar{z}_0)\left(p+\frac{1}{2}\right)(-z_0)^{-p-3/2} + 2(z_0+\bar{z}_0)\left(p+\frac{1}{2}\right)^2(-z_0)^{-p-5/2} + \varkappa^2(-\bar{z}_0)^{-p-1/2} +$$
$$+ 2\varkappa\bar{z}_0\left(p+\frac{1}{2}\right)(-\bar{z}_0)^{-p-3/2}.$$

Introducing $z_0 = R_0 e^{i\alpha}$ $(0 < \alpha < 1/2\pi)$, from equation (11) we obtain

$$R(p) = \frac{2i \operatorname{tg} \pi p T(p)}{2\varkappa \sin \pi p - \varkappa^2 + 4p(p+1)} R_0^{-p-1/2} \tag{12}$$

where

$$T(p) = 2Q\varkappa \sin\left[\pi p - \alpha\left(p+\frac{1}{2}\right)\right] +$$
$$+ Q\left\{-2\left(p+\frac{1}{2}\right)\sin\alpha \sin\left[\pi p - \alpha\left(p+\frac{3}{2}\right)\right] - \cos\left(p+\frac{1}{2}\right)\alpha +\right.$$
$$\left.+ 2\left(p+\frac{1}{2}\right)\left[2\left(p+\frac{1}{2}\right)-\varkappa\right]\cos\alpha\cos\left(p+\frac{3}{2}\right)\alpha + \varkappa\left[2\left(p+\frac{1}{2}\right)-\varkappa\right]\cos\left(p+\frac{1}{2}\right)\alpha\right\} +$$
$$+ P\left\{2\left(p+\frac{1}{2}\right)\sin\alpha\sin\left[\pi p - \alpha\left(p+\frac{3}{2}\right)\right] - \sin\left(p+\frac{1}{2}\right)\alpha +\right.$$
$$\left.+ 2\left(p+\frac{1}{2}\right)\left[2\left(p+\frac{1}{2}\right)-\varkappa\right]\cos\alpha\sin\left(p+\frac{3}{2}\right)\alpha - \varkappa\left[2\left(p+\frac{1}{2}\right)-\varkappa\right]\sin\left(p+\frac{1}{2}\right)\alpha\right\}$$

Applying inverse transformation, we find that

$$\psi(\xi) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{2i \operatorname{tg} \pi p T(p)}{2\varkappa \sin \pi p - \varkappa^2 + 4p(p+1)} R_0^{-p-1/2} e^{p\xi} dp \quad \left(-\frac{1}{2} < \sigma < 0\right) \tag{13}$$

Introducing $x = e^{\xi}$, $r(x) = \psi(\xi)$, and referring to equation (7), taking into consideration that

$$\frac{1}{\pi}\int_0^\infty \frac{t^p}{t-x}\,dt = -\frac{x^p}{\operatorname{tg}\pi p}$$

we obtain

$$q(x) = -\frac{1}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{R_0^{-p-1/2}x^{p-1/2}T(p)}{2\varkappa\sin\pi p - \varkappa^2 + 4p(p+1)}\,dp$$

It is convenient to introduce $s = p + 1/2$ as the variable of integration. Then, as a final result, we will have

$$q(x) = -\frac{1}{\pi i x}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{S(s)}{4s^2 - 2\varkappa\cos\pi s - (1+\varkappa^2)}\left(\frac{x}{R_0}\right)^s ds \qquad \left(0<\gamma<\tfrac{1}{2}\right) \quad (14)$$

where

$$\begin{aligned} S(s) = &-2Q\varkappa\cos(\pi-\alpha)s + Q\{-2s\sin\alpha\sin[\pi s - \alpha(s+1)] - \cos\alpha s + \\ &+ 2s(2s-\varkappa)\cos\alpha\cos(s+1)\alpha + \varkappa(2s-\varkappa)\cos\alpha s\} + \\ &+ P\{-2s\sin\alpha\cos[\pi s - \alpha(s+1)] - \sin\alpha s + \\ &+ 2s(2s-\varkappa)\cos\alpha\sin(s+1)\alpha - \varkappa(2s-\varkappa)\sin\alpha s\} \end{aligned}$$

While computing integrals, when $x < R_0$, the calculations are taken from the right, and when $x > R_0$ from the left side of the straight line γ. In particular, when $x < R_0$, we have

$$q(x) = \frac{1}{x}\sum_k \left(\frac{x}{R_0}\right)^{\rho_k}\left[\operatorname{Re}\Omega_k\cos\left(\theta_k\ln\frac{x}{R}\right) - \operatorname{Im}\Omega_k\sin\left(\theta_k\ln\frac{x}{R}\right)\right] \qquad (15)$$

where

$$\Omega_k = \frac{S(s_k)}{\varkappa\pi\sin\pi s_k + 4s_k}, \qquad s_k = \rho_k + i\theta_k \qquad \left(\rho_k > 0,\ 0<\theta_k<\tfrac{1}{2}\pi\right)$$

and S_k are the roots of equation

$$4s^2 - 2\varkappa\cos\pi s - (1+\varkappa^2) = 0 \qquad (16)$$

As equation (16) always has a root for which $\rho < 1$, it is possible to draw the conclusion that when $X_y = 1/2\ q(x)$, a corner of the elastic quadrant is approached, the stress, in absolute value, keeps increasing to infinity, while simultaneously changing its sign an infinite number of times.

If we assume that $s = 2\lambda + 1$, then equation (16) will coincide with the equation for determination of order of stress increase in the proximity of the angle. The latter equation was obtained in paper [1].

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